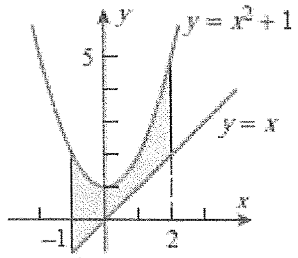
HOMEWORK 2 *Model Answer.* **Calculus (2)**

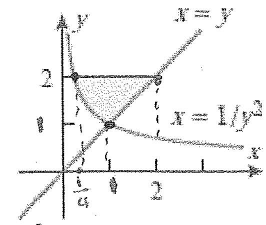
First Semester 2015/2016

Question(1): Find the area of the shaded region



$$A = \int_{-1}^{2} x^2+1-x \cdot dx = \frac{x^3}{3} + x - \frac{x^2}{2}\Big]_{-1}^{2}$$

$$= \frac{9}{2}$$



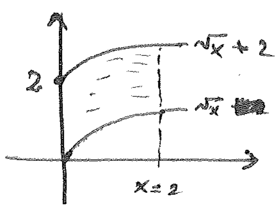
$$A = \int_{1}^{2} y - \frac{1}{y^2} \cdot dy = \frac{y^2}{2} + \frac{1}{y}\Big]_{1}^{2} = 1$$

OR

$$A = \int_{1/4}^{1} 2 - \frac{1}{\sqrt{x}} \cdot dx + \int_{1}^{2} 2 - x \cdot dx = 1$$

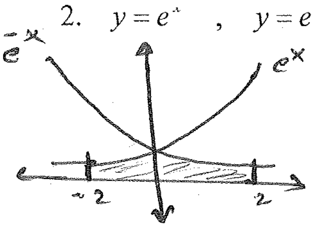
Question(2): Find the area of the region that is enclosed by the curves

1. $y=\sqrt{x}$, $y=\sqrt{x}+2$, $x=0$ and $x=2$



$$A = \int_{0}^{2} \sqrt{x} - \sqrt{x} + 2 \cdot dx = 2x\Big]_{0}^{2} = 4$$

2. $y=e^{x}$, $y=e^{-x}$, $y=0$ on $[-2,2]$

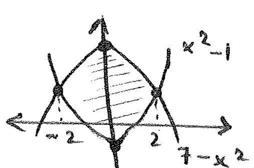


$$A = \int_{-2}^{0} e^{x} \cdot dx + \int_{0}^{2} e^{-x} \cdot dx = e^{x}\Big]_{-2}^{0} + -e^{-x}\Big]_{0}^{2}$$

$$= 1 - e^{-2} + -e^{-2} + 1$$

$$= 2 - 2e^{-2}.$$

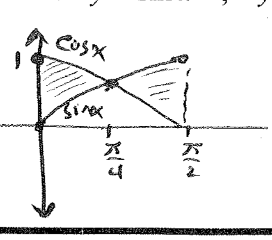
3. $y=x^2-1$, $y=7-x^2$



$$A = \int_{-2}^{2} 7 - x^2 - x^2 + 1 \cdot dx = \int_{-2}^{2} 8 - 2x^2 \cdot dx = 32 - \frac{32}{3}$$

$$= \frac{64}{3}$$

4. $y=\sin x$, $y=\cos x$ on $\left[0,\frac{\pi}{2}\right]$



$$A = \int_{0}^{\pi/4} \cos x - \sin x \cdot dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \cdot dx$$

$$= \sqrt{2} - 1 + \sqrt{2} - 1 = 2\sqrt{2} - 2$$

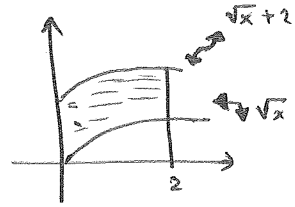
Question(3): Find the volume of the solid that results when the region enclosed by the given curves is revolved about ....

1. $y=\sqrt{x}$, $y=\sqrt{x}+2$, $x=0$ and $x=2$ (about the x–axis)

$$V = \pi \int_0^2 (\sqrt{x}+2)^2 - (\sqrt{x})^2 . dx = \pi \int_0^2 x + 4x^{1/2} + 4 - x . dx$$

$$= \pi \int_0^2 4x^{1/2} + 4 . dx = \pi \left[\frac{8x^{3/2}}{3} + 4x \right]_0^2$$

$$= \pi \left[8 + \frac{16\sqrt{2}}{3} \right]$$

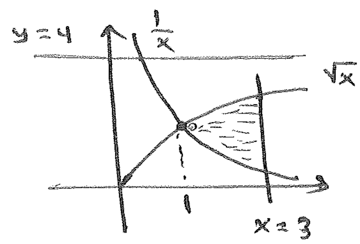


2. $y=\frac{1}{x}$, $y=\sqrt{x}$, $x=3$ (about the line y = 4)

$$V = \pi \int_1^3 \left(4-\frac{1}{x}\right)^2 - (4-\sqrt{x})^2 . dx$$

$$= \pi \left[\int_1^3 \cancel{16} - \frac{8}{x} + \frac{1}{x^2} - \cancel{16} + 8\sqrt{x} - x . dx \right]$$

$$= \pi \left[-4 + 16\sqrt{3} - 8\ln 3 \right]$$

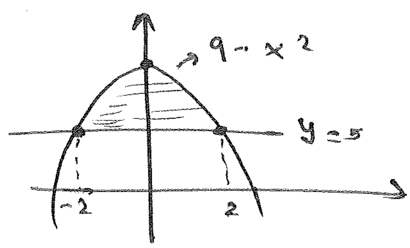


3. $y=9-x^2$, $y=5$ (about the line y = −1)

$$V = \pi \int_{-2}^{2} (9-x^2+1)^2 - (5+1)^2 . dx$$

$$= \pi \int_{-2}^{2} 100 - 20x^2 + x^4 - 36 . dx$$

$$= \frac{2432}{15} \pi$$



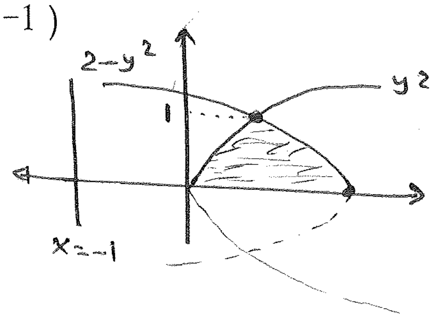
4. $y=\sqrt{x}$, $y=\sqrt{2-x}$, $y=0$ (about the line x = −1)

$\Downarrow$ $x=y^2$; $\Downarrow$ $y^2 = 2-x$, $x = 2-y^2$

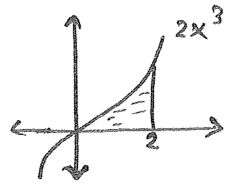
$$V = \pi \int_0^1 (2-y^2+1)^2 - (y^2+1)^2 . dy$$

$$= \pi \int_0^1 9 - 6y^2 + \cancel{y^4} - \cancel{y^4} - 2y^2 - 1 . dy$$

$$= \pi \int_0^1 8 - 8y^2 . dy = \frac{16}{3}\pi$$



Question(4): Find the volume of the solid whose base is the region bounded between the curve $y = 2x^3$ and the x-axis from $x=0$ to $x=2$ and whose cross sections taken perpendicular to the x-axis are squares.



$$A(x) = L^2 = (2x^3)^2 = 4x^6$$

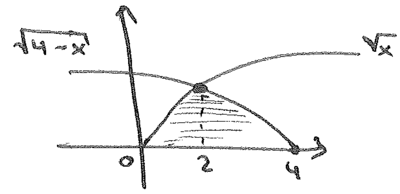
$$V = \int_0^2 4x^6 \, dx = \left.\frac{4x^7}{7}\right]_0^2$$

$$= \frac{512}{7}$$

Question(5): Use cylindrical shells to find the volume of the solid generate when the region enclosed by the given curves is revolved about the y-axis

1. $y=\sqrt{x}$, $y=\sqrt{4-x}$, $y=0$

$\rightarrow u = 4-x$, $du = -dx$



$$V = 2\pi \int_0^2 x \cdot \sqrt{x} \, dx + 2\pi \int_2^4 x \cdot \sqrt{4-x} \, dx$$

$$= 2\pi \left[\frac{2x^{5/2}}{5}\right]_0^2 + {}^{-}2\pi \int_2^0 (4-u) \cdot u^{1/2} \, du$$

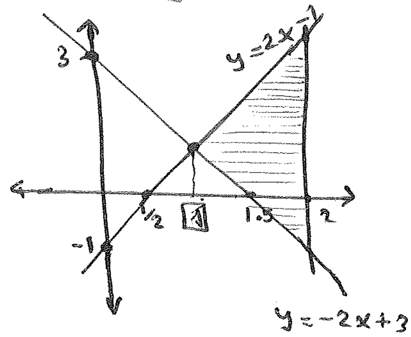
$\sqrt{4-x} = \sqrt{x}$

$4 - x = x \Rightarrow x = 2$

$$= 2\pi \left[\frac{2x^{5/2}}{5}\right]_0^2 + {}^{-}2\pi \left[\frac{4u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right]_2^0 = 2\pi\left[\frac{8\sqrt{2}}{5} - \frac{56\sqrt{2}}{15}\right]$$

$$= \frac{32\sqrt{2}}{3}\pi$$

2. $y = 2x-1$, $y=-2x+3$, $x=2$



$$V = 2\pi \int_1^2 x \cdot (2x - 1 + 2x - 3) \cdot dx$$

$$= 2\pi \int_1^2 x \cdot (4x - 4) \cdot dx$$

$$= 2\pi \left[\frac{4x^3}{3} - \frac{4x^2}{2}\right]_1^2 = \frac{20\pi}{3}$$

3. $y = \cos(x^2)$, $y=0$, $x=0$, $x = \frac{\sqrt{\pi}}{2}$

$$V = 2\pi \int_0^{\sqrt{\pi}/2} x \cdot \cos(x^2) \cdot dx$$

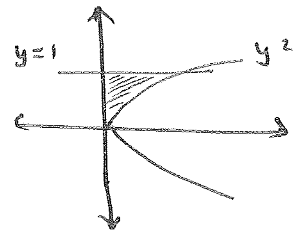
$u = x^2$

$du = 2x \cdot dx$

$$= 2\pi \int_0^{\sqrt{\pi}/2} x \cdot \cos u \cdot \frac{du}{2x} = \pi \left[\sin x^2\right]_0^{\sqrt{\pi}/2} = \frac{\pi}{\sqrt{2}}$$

Question(6): Use cylindrical shells to find the volume of the solid generate when the region enclosed by the given curves is revolved about the x–axis

1. $y^2 = x$, $y = 1$, $x = 0$



$$V = 2\pi \int_0^1 y \cdot y^2 \cdot dy = 2\pi \left[\frac{y^4}{4}\right]_0^1$$

$$= \frac{\pi}{2}$$

2. $yx = 4$, $x + y = 5$ $\Rightarrow \frac{4}{y} = 5 - y \Rightarrow 4 = 5y - y^2$

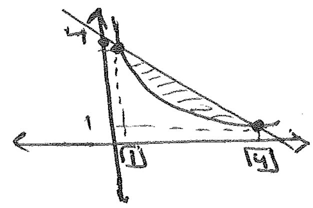
$x = \frac{4}{y}$ $\quad$ $x = 5 - y$

$$y^2 - 5y + 4 = 0$$
$$(y-1)(y-4) = 0$$
$$y = 1, \quad y = 4$$

$$V = 2\pi \int_1^4 y \cdot \left(5 - y - \frac{4}{y}\right) \cdot dy$$

$$= 2\pi \int_1^4 5y - y^2 - 4 \cdot dy = 2\pi \left[\frac{5y^2}{2} - \frac{y^3}{3} - 4y\right]_1^4$$

$$= 9\pi$$



Question(7): Find the exact arc length of the following curve

$24yx = y^4 + 48$ from y = 2 to y = 4. $\Rightarrow x = \frac{y^3}{24} + \frac{2}{y}$

$$x' = \frac{\cancel{3}y^2}{\cancel{24}\,8} - \frac{2}{y^2}$$

$$(x')^2 = \frac{y^4}{64} - \frac{1}{2} + \frac{4}{y^4}$$

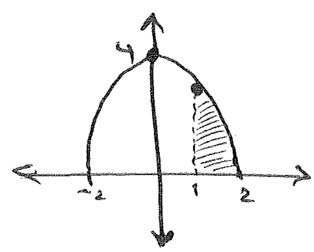
$$L = \int_2^4 \sqrt{1 + \frac{y^4}{64} - \frac{1}{2} + \frac{4}{y^4}} \cdot dy$$

$$= \int_2^4 \sqrt{\frac{y^4}{64} + \frac{1}{2} + \frac{4}{y^4}} \cdot dy$$

$$= \int_2^4 \sqrt{\left(\frac{y^2}{8} + \frac{2}{y^2}\right)^2} \cdot dy = \int_2^4 \frac{y^2}{8} + \frac{2}{y^2} \cdot dy = \frac{17}{6} = 2.83$$

Question(8): Find the area of the surface generated by revolving the given curve about the x–axis.

1. $y = \sqrt{4 - x^2}$, $\boxed{1 \le x \le 4}$ $\qquad y' = \frac{-2x}{2\sqrt{4 - x^2}}$



$$S = 2\pi \int_1^2 \sqrt{4 - x^2} \cdot \sqrt{1 + \frac{4x^2}{4(4 - x^2)}} \cdot dx$$

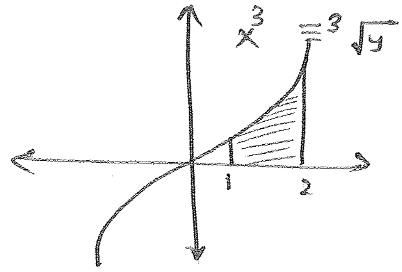
$$= 2\pi \int_1^2 \sqrt{4 - x^2} \cdot \sqrt{\frac{4 - x^2 + x^2}{4 - x^2}} \cdot dx = 2\pi \int_1^2 2 \cdot dx$$

$$= 2\pi \left[2x\right]_1^2 = 4\pi .$$

2. $x=\sqrt[3]{y}$, $1\le y\le 8$ $\Rightarrow$ $y'=3x^2$

$(y')^2=9x^4$

$y=x^3$, $1\le x\le 2$



$$S=2\pi\int_1^2 x^3\cdot\sqrt{1+9x^4}\cdot dx$$, $u=1+9x^4$,

$du=36x^3\cdot dx$

$$=2\pi\int_{10}^{145}\cancel{x^3}\cdot\sqrt{u}\cdot\frac{du}{36\cancel{x^3}}$$

$$=\frac{2\pi}{36}\cdot\left[\frac{u^{3/2}}{3/2}\right]_{10}^{144}=\frac{5}{27}\left(29\sqrt{145}-2\sqrt{10}\right)\pi$$

Question(9): Find the area of the surface generated by revolving the given curve about the y–axis.

1. $x=\sqrt{9-y^2}$, $-2\le y\le 2$, $x'=\frac{-\cancel{2}y}{\cancel{2}\sqrt{9-y^2}}$ $\Rightarrow (x')^2=\frac{y^2}{9-y^2}$

$$S=2\pi\int_{-2}^{2}\sqrt{9-y^2}\cdot\sqrt{1+\frac{y^2}{9-y^2}}\cdot dy$$

$$=2\pi\int_{-2}^{2}3\cdot dy=24\pi\ .$$

1. $x=9y+1$, $0\le y\le 2$, $x'=9$ $\Rightarrow$ $(x')^2=81$

$$S=2\pi\int_0^2(9y+1)\cdot\sqrt{82}\cdot dy$$

$$=2\sqrt{82}\,\pi\left[\frac{9y^2}{2}+y\right]_0^2=40\sqrt{82}\,\pi$$